CONDITIONS ON THE EQUATION OF STATE FOR A VISCOELASTIC MEDIUM

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INTRODUCTION

Godunov and Kozin [1] investigated the structure of shock waves in a viscoelastic medium characterized by a relaxation time τ of tangential stresses and an equation of the medium of special form. The form of the equation of state was dictated by considerations of convenience of the construction of interpolation formulas. In the present paper restrictions on the equation of state of general form are formulated.

1. Equation of Elastic Energy

We consider an isotropic medium with an internal energy density given by the equation

$$E = \hat{E}(k_1, k_2, k_3, S), \tag{1.1}$$

where $\hat{E}$ is a symmetric function of the specific elongations $k_i > 0$ ($i = 1, 2, 3$) along the principal axes of strain, and S is the entropy per unit mass. We assume that (1.1) satisfies the following conditions:

$$T = \hat{E}_s > 0, \ \hat{r} = \frac{k_1\hat{E}_{k_1} - k_2\hat{E}_{k_2}}{k_1 - k_2} > 0; \tag{1.2}$$

$$\left.\begin{aligned} &c^2 = \hat{E}_{k_1^2} > 0, \ l = \hat{E}_{k_1 S} < 0, \\ &g = k_1 k_2 \hat{E}_{k_1 k_2} - k_1^2 \hat{E}_{k_1^2} - k_1 \hat{E}_{k_1} = \frac{k_1 \hat{E}_{k_1 S}}{\hat{E}_S}(k_1 E_{k_1} - k_2 E_{k_2}) < 0; \end{aligned}\right\} \tag{1.3}$$

$$\hat{q} = \frac{\partial c^2}{\partial k_1} = \hat{E}_{k_1^3} < 0, \ \frac{1}{3} k_1^2 \hat{E}_{k_1^2} + \frac{2}{3} k_1 k_2 \hat{E}_{k_1 k_2} + \frac{1}{3} k_1 \hat{E}_{k_1} > 0. \tag{1.4}$$

Since $\hat{E}$ is symmetric along k_i, similar inequalities can be obtained from (1.2)-(1.4) by cyclic permutation of the k_i. As a consequence of inequalities (1.2)-(1.4) and the symmetry of $\hat{E}$ there are inequalities for $\hat{E}(k_1, k_2, k_3, S)$ for uniform elongations in all directions, i.e., $k_1 = k_2 = k_3 = k$. These inequalities have the form

$$\hat{E}_{vv} > 0, \ \hat{E}_{vS} < 0, \ \hat{E}_{vvv} < 0, \ k = \sqrt[3]{v/v^0}. \tag{1.5}$$

Henceforth the internal energy E will be assumed to be a function of the parameters α, β, and γ, representing the logarithms of the specific elongations along the principal axes of strain

$$\alpha = \ln k_1, \ \beta = \ln k_2, \ \gamma = \ln k_3, \ E = E(\alpha, \beta, \gamma, S). \tag{1.6}$$

In this case inequalities (1.2)-(1.4) have the form

$$T = E_s > 0, \ r = (E_\alpha - E_\beta)/(\alpha - \beta) > 0; \tag{1.7}$$

$$\left.\begin{aligned} &c^2 = E_{\alpha\alpha} - E_\alpha > 0, \ l = E_{\alpha S} < 0, \\ &g = E_{\alpha\beta} - E_{\alpha\alpha} - E_{\alpha S}(E_\beta - E_\alpha)/E_S < 0; \end{aligned}\right\} \tag{1.8}$$

$$q = E_{\alpha\alpha\alpha} - 3E_{\alpha\alpha} + 2E_\alpha < 0, \ (1/3) E_{\alpha\alpha} + (2/3) E_{\alpha\beta} - E_\alpha > 0. \tag{1.9}$$

2. Structure of Shock Waves

The differential equations describing the motion of a viscoelastic medium parallel to the x axis in (x, y, z) space have the form [1]

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$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}=0,\ \frac{\partial\rho u}{\partial t}+\frac{\partial(\rho u^2-\sigma_x)}{\partial x}=0,$$
$$\frac{\partial\rho(E+u^2/2)}{\partial t}+\frac{\partial[\rho u(E+u^2/2)-\sigma_x u]}{\partial x}=0,\qquad (2.1)$$
$$\frac{\partial\beta}{\partial t}+u\frac{\partial\beta}{\partial x}=-\frac{\beta-(\alpha+\beta+\gamma)/3}{\tau},$$

where x and t are space and time coordinates, u is the velocity of the medium along the x axis, and $\rho=\rho_0\cdot e^{-\alpha-\beta-\gamma}$. Because of the isotropy of the medium $\beta\equiv\gamma$. The quantity ρ_0 is the density of the medium under normal conditions. The principal stresses σ_x, σ_y, and σ_z along the x, y, and z axes are related to the strains by the expressions

$$\sigma_x=\rho E_\alpha,\ \sigma_y=\rho E_\beta,\ \sigma_z=\rho E_\gamma.$$

As shown in [1], the problem of the structure of shock waves [solutions of system (2.1) of the form $\alpha=\alpha(x-Ut)$, $\beta=\gamma=\beta(x-Ut)$, $S=S(x-Ut)$] reduces to the problem of solving the equations

$$\sigma_x+p_0=w^2(v-v_0),\ E-E_0+[(p_0-\sigma_x)/2](v-v_0)=0,\ \beta=\gamma \qquad (2.2)$$

and the quadrature

$$dx/d\beta=3\tau w/\rho(\alpha-\beta), \qquad (2.3)$$

where p_0, ρ_0, E_0, and u_0 characterize the state of the material in front of the wave, $w=\rho_0u_0$ is the mass velocity, and $v=1/\rho$. The characteristic relaxation time τ of stresses resulting from plastic deformations depends on the state of stress of the medium.

System (2.2) determines a curve in (α,β,S) space which we call, as in [1], the curve of possible states. Calculations performed in [1] for equations of state of the form

$$E=E(v,D,S),\ D=\frac{1}{2}(d_1^2+d_2^2+d_3^2),\ d_1=\alpha-(\alpha+\beta+\gamma)/3,$$
$$d_2=\beta-(\alpha+\beta+\gamma)/3,\ d_3=\gamma-(\alpha+\rho+\gamma)/3$$

showed that the curve of possible states is a smooth curve connecting the states $(\alpha_0=\beta_0, S_0)$ and $(\alpha_1=\beta_1, S_1)$ in front of and behind the wave, respectively.

For $M_0=|w|/\rho_0c_0>1$, where c_0 is the speed of sound (1.8), the shock wave contains an elastic jump (precursor) which is determined by the subsidiary relation $\beta=\beta_0$ and is located in front of the plastic wave arising on the profile as a result of the nonlinear dependence on the material parameters. Calculations showed that the entropy S and the value of β on the wave increase monotonically. It will be shown below that the profile has a similar structure for media with an equation of state (1.6)-(1.9), which means also for (1.1)-(1.4).

3. Properties of the Curve of Possible States

Let us consider the functions $s=(p-p_0)/(v_0-v)$, $H=E-E_0+(p+p_0)(v-v_0)/2$, $p=-\sigma_x=-\rho E_\alpha$, $v=v_0e^{\alpha+\beta+\gamma}$.

The differentials of these functions have the form

$$dv=v(d\alpha+d\beta+d\gamma),$$
$$dp=-\rho^2c^2dv+\rho(E_{\alpha\beta}-E_{\alpha\alpha})d\beta+\rho(E_{\alpha\gamma}-E_{\alpha\alpha})d\gamma+\rho E_{\alpha S}dS,$$
$$ds=[(v_0-v)dp+(p-p_0)dv]/(v-v_0)^2,$$
$$dH=TdS+(E_\beta-E_\alpha)d\beta+(E_\gamma-E_\alpha)d\gamma-(1/2)(v-v_0)^2ds.$$

We write the curve of possible states (2.2), which is given by the equation $H=0$, $s=w^2$, $\beta=\gamma$, in parametric form $\alpha=\alpha(v)$, $\beta=\beta(v)$, $S=S(v)$. Then the differential equations of the curve of possible states have the form

$$\frac{d\alpha}{dv}=\rho-\frac{\rho c^2}{g}(M^2-1),\ \alpha(v_0)=\ln\sqrt[3]{v_0/v^0},$$
$$\frac{d\beta}{\partial v}=\frac{\rho c^2}{g}(M^2-1),\ \beta(v_0)=\ln\sqrt[3]{v_0/v^0}, \qquad (3.1)$$
$$\frac{dS}{dv}=\frac{\rho c^2}{T}(E_\beta-E_\alpha)(1-M^2),\ S(v_0)=S_0,$$

where the Mach number $M=|w|/\rho c$. It follows from (3.1) that along the curve of possible states

$$d\rho^2c^2/dv=\rho^3[q+(c^2(M^2-1)/g)(E_{\alpha\alpha\beta}-E_{\alpha\alpha\alpha}-E_{\alpha\beta}+E_{\alpha\alpha}-(E_\beta-E_\alpha)(E_{\alpha\alpha S}-E_{\alpha S}))]. \qquad (3.2)$$

Let us establish certain properties of the curve of possible states.

A. The state of the material behind and in front of the wave satisfies the following conditions [1]:

$$\left.\begin{aligned} &E_1 - E_0 + (p_1 + p_0)(v_1 - v_0)/2 = 0,\ p_1 - p_0 = -w^2(v_1 - v_0),\\ &\alpha_1 = \beta_1 = \gamma_1 = \ln\sqrt[3]{v_1/v^0},\ \alpha_0 = \beta_0 = \gamma_0 = \ln\sqrt[3]{v_0/v^0} \end{aligned}\right\} \quad (3.3)$$

the subscripts 1 and 0 correspond to the beginning and end of the wave. Since Eqs. (3.3) are the ordinary gas-dynamic relations, conditions (1.5) are sufficient to ensure a unique solution of (3.3) $(v_1, S_1) \neq (v_0, S_0)$ for given values of v_0, S_0, and w [2]. This implies that the curve of possible states has two and only two points of intersection with the plane $\alpha = \beta$ in (α, β, S) space corresponding to the initial and final states of the shock wave for a given w.

B. We call the point $Z^* = (\alpha^*, \beta^*, S^*)$ on the curve of possible states at which M=1, i.e., $|w| = \rho c$, the critical point. It follows from (3.2) that at the critical point

$$d(\rho^2c^2)/dv = \rho^3 q < 0,\ dM^2/dv > 0, \quad (3.4)$$

i.e., the value of $M^2 - 1$ is increasing. It follows from (3.1) that $d\beta/dv = dS/dv = 0$ at $M^2 = 1$; i.e., $\beta(v)$ and $S(v)$ at the critical point cease to be monotonic functions.

C. We consider a portion of the curve of possible states for the interval $v_- < v < v_+$. If $M^2(v_+) < 1$, $M^2(v) < 1$ for the whole interval (v_-, v_+). If this were not the case, there would be a point v^*, $v_- \leq v^* \leq v_+$, at which $M^2(v^*) = 1$, $dM^2(v^*)/dv < 0$, which would contradict (3.4).

D. Finally, the curve of possible states is a simple curve in (α, β, S) space; i.e.,

$$0 < (d\alpha/dv)^2 + (d\beta/dv)^2 + (dS/dv)^2 < \infty. \quad (3.5)$$

By virtue of (3.1) the violation of inequality (3.5) would contradict (1.7) and (1.8).

4. Construction of the Wave Profile

We consider first the case $M_0 = |w|/\rho_0 c_0 < 1$, i.e., the subsonic case of the propagation of a shock wave. Let $v_0 > v_1$ be the values of v corresponding to the beginning and end of the shock wave. Since $M_0{}^2 = M^2(v_0) < 1$, as shown in case C, $M^2 < 1$ everywhere in the interval $v_1 < v < v_0$. According to (3.1) and (1.8) this implies that $d\beta/dv > 0$, i.e., $\beta(v)$ is a monotonically increasing function. For $v_1 < v < v_0$ a portion of the curve of possible states lies on one side of the plane $\beta = \alpha$ in (α, β, S) space (case A). Following [1] we expand the curve of possible states in the neighborhood of the point $(\alpha_0 = \beta_0, S_0)$. We obtain

$$\beta - \alpha \approx \frac{1 - M_0^2 + \frac{2}{3c_0^2}(E^0_{\alpha\beta} - E^0_{\alpha\alpha})}{1 - M_0^2}\left(\frac{d\beta}{dv}\right)_0 (v - v_0),\ S \approx S_0.$$

It follows from (1.9) that $^2/_3\,(E^0_{\alpha\beta} = E^0_{\alpha\alpha})/c_0{}^2 + 1 = a_0^2/c_0{}^2 > 0$. Since $M_0{}^2 > a_0^2/c_0^2$, $(d\beta/dv)_0 > 0$, it follows that $\beta - \alpha \geq 0$ in the neighborhood of $v \lessgtr v_0$, which means that $\beta \geq \alpha$ everywhere in the interval $v_1 \leq v \leq v_0$. We conclude from (3.1) and (1.7) that $dS/dv \leq 0$ for $v_1 \leq v \leq v_0$; i.e., S(v) is a monotonic function. Under these conditions the quadrature (2.3) determines the parameter $x = x(\beta)$ as a monotonic function of β, which means x(v) is also a monotonic function.

We turn to the supersonic case when $M_0 = |w|/\rho_0 c_0 > 1$. As shown in [1] it is impossible to construct a smooth solution for the shock wave in this case. The same is true also for an equation of state $E(\alpha, \beta, \gamma, S)$ of general form. Following [1] we introduce an elastic jump from the initial state $(\alpha_0 = \beta_0, S_0)$ to an intermediate state (α_2, β_2, S_2) defined by the conditions

$$\left.\begin{aligned} &E_2 - E_0 + [(p_2 + p_0)/2](v_2 - v_0) = 0,\\ &p_2 = p_0 - w^2(v_2 - v_0),\ \gamma_2 = \beta_2 = \beta_0. \end{aligned}\right\} \quad (4.1)$$

The inequalities (1.7)-(1.9) [2] are sufficient to ensure that (4.1) has a unique solution (α_2, β_2, S_2) and in this case $M_2 = |w|/\rho_2 c_2$ would be smaller than unity.

Let us consider now the portion of the curve of possible states $\alpha(v)$, $\beta(v)$, $S(v)$, where $v_1 \leq v \leq v_2$. Since $M_2^2 = M^2(v_2) > 1$, the statements formulated for the subsonic case are valid for this portion; i.e., $M^2 < 1$ and $\beta(v)$, S(v), and x(v) are monotonic functions everywhere in the interval $v_1 \leq v \leq v_2$. This permits the continuous extension of the solution for a shock wave beyond the elastic jump.

5. Inequalities for an Equation of State of the Form E(v, D, Δ, S)

It is of interest to reformulate the inequalities (1.7)-(1.9) for an equation of state E = E(v, D, Δ, S), which is frequently used in applications $[\Delta = {}^1/_3 (d_1^3 + d_2^3 + d_3^3)]$. However, the relaxation of tangential stresses as a result of plastic deformations, characterized by a relaxation time τ, occurs more rapidly the larger the value of the tangential stresses, i.e., the larger the values of D and Δ. As a result of the essentially nonlinear character of the dependence of τ on the tangential stresses [1], in actual processes

$$|d_1| + |d_2| + |d_3| \ll 1,\ D \ll 1,\ |\Delta| \ll 1.$$

Under these conditions by neglecting the terms containing d_i as factors, inequalities (1.7)-(1.9) can be written in the form

$$\left.\begin{array}{l} r = E_D > 0,\ c^2 = v^2 E_{vv} + (2/3) E_D > 0, \\ l = vE_{vS} < 0,\ T = E_S > 0,\ g = -E_D < 0, \\ q = -2E_D + 2vE_{vD} + v^2 E_{vvv} + (4/3) E_\Delta < 0,\ a^2 = v^2 E_{vv} > 0. \end{array}\right\} \quad (5.1)$$

The inequalities

$$E_D > 0,\ E_{vv} > 0,\ E_{vS} < 0,\ E_{vD} < 0,\quad E_{vvv} < 0,\ E_\Delta < 0,$$

which are satisfied for interpolation formulas of the equations of state E(v, D, S) given in [3], are sufficient to ensure that (5.1) are satisfied.

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SHOCK ADIABATS OF ALKALI HALIDE CRYSTALS

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Nonparametric calculation of shock adiabats makes it possible to relate shock-compression parameters of a material directly determinable in experiment with parameters characterizing the material on an atomic level. Establishment of such a relationship is a necessary step in preliminary calculation of shock-compression parameters, which are of great significance in planning experiments and in problems involving construction with materials having given optimum properties.

Nonparametric calculation of shock adiabats of alkali halide crystals is of interest because these crystals have been studied experimentally in great detail, allowing experimental verification of calculations. At the same time, if we consider that many inorganic materials, including sitalls, glasses, ceramics, and some explosives, have ionic or predominantly ionic bonds, study of alkali halide crystals is necessary to be able to consider the behavior of these materials under shock-compression conditions.

The shock adiabat $P_\Gamma(V)$ can be calculated with the formula [1]

$$P_\Gamma(V) = \frac{P_x(V) + \gamma(V)[E_0 - U(V)]/V}{1 + \gamma(V)[1 - V_0/V]/2}, \quad (1)$$

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